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Supersymmetrzation of Harmonic Oscillator

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ABSTRACT: In this paper we have constructed a simple supersymmetric quantum mechanical system that include the bosonic oscillator degree of freedom $\left(\hat{a}^{\dagger}, \hat{a}\right)$ and fermionic spin $\left(-\frac{1}{2}\right)$ degrees of freedom $\left(\hat{b}^{\dagger}, \hat{b}\right)$ and call it supersymmetric harmonic oscillator. The supersymmetry is obtained by annihilating simultaneously one bosonic quantum $n_b \to n_b - 1$ and creating one fermionic quantum $n_f \to n_f - 1$ or vice versa. The supercharges have been represented by bosonic and fermionic operators for position and momentum operators and accordingly we have obtained the supersymmetric Hamiltonian operator. As such we have calculated the energy of supersymmetric harmonic oscillator, which shows that SUSY does not break for ground state.

Keywords: Supersymmetric quantum mechanical system, bosonic and fermionic operators, supersymmetric Hamiltonian operator

I. INTRODUCTION

Supersymmetry was first appeared in field theories in terms of bosonic and fermionic fields and the possibility was early observed that it could accommodate a Grand Unified Theory (GUT) for four basic interactions of nature (strong, weak, electromagnetic and gravitational). Gelfand and Liktman [1] did the first work on super algebra in space-time within the framework of Poincare algebra. The SUSY algebra in quantum mechanics was initiated within the work of Nicolai [2]. The bosonic degrees of freedom are characterized by bosonic creation and annihilation operators obeying the commutation relations. Similarly fermionic degrees of freedom are described by fermionic creation and annihilation operators and obey anticommutation relations. Nicolai's SUSY algebra was described as N = 1 SUSY algebra, has been extended by Witten [3].
The exact supersymmetry describes symmetry between bosonic and fermionic degree of freedom and is essential ingredient in grand unified theory. The structure of Lie algebra that incorporates commutation and anticommutation relations, characterizes a new type of symmetry, dynamical symmetry which is supersymmetry i.e. symmetry that converts bosonic part into fermionic part and fermionic part into bosonic part. Hamiltonian is one of the generator of this super algebra, remains invariant under such transformations. So that tremendous physical contents are included in it as it connects different quantum systems. In quantum mechanical system SUSY has been found to be very useful [4].
We have constructed a simple supersymmetric quantum mechanical system that include the bosonic oscillator degree of freedom $\left(\hat{a}^{\dagger}, \hat{a}\right)$ and fermionic spin $\left(-\frac{1}{2}\right)$ degrees of freedom $\left(\hat{b}^{\dagger}, \hat{b}\right)$ and call it supersymmetric harmonic oscillator. The supersymmetry is obtained by annihilating simultaneously one bosonic quantum $n_b \to n_b - 1$ and creating one fermionic quantum $n_f \to n_f - 1$ or vice versa. The supercharges have been represented by bosonic and fermionic operators for position and momentum operators and accordingly we have obtained the supersymmetric Hamiltonian operator. These supercharges represents conversion of a fermionic state to a bosonic state and bosonic state to fermionic state. Supercharges always commute with usual Hamiltonian. Thus the anticommuting charges in quaternion formalism combine to form the generators of time translation, namely the Hamiltonian $\hat{H}$. The ground state of this system is the state $\left|0\right\rangle_{osc}\left|0\right\rangle_{spin}$ or $\left|0\right\rangle_{boson}\left|0\right\rangle_{fermion} = \left|0,0\right\rangle$, where both bosonic and fermionic degrees of freedom are in the lowest energy state. As such we have calculated the energy of supersymmetric harmonic oscillator, which shows that SUSY does not break for ground state. Thus the quaternion reformulationation of a super symmetry gives rise to a simple representation of super symmetry in quantum mechanics. It is however, trivial since it describes non-interacting boson (oscillator) and fermions (spin $\left(-\frac{1}{2}\right)$ particles).

II. SUPERSYMMETRIC HARMONIC OSCILLATOR

Let us now construct a simple supersymmetric quantum mechanical system that include the bosonic oscillator degree of freedom $\left(\hat{a}^{\dagger}, \hat{a}\right)$ and fermionic spin $\left(-\frac{1}{2}\right)$ degrees of freedom $\left(\hat{b}^{\dagger}, \hat{b}\right)$ .We call it as supersymmetric harmonic oscillator .The supersymmetry is obtained by annihilating simultaneously one bosonic quantum $n_b \to n_b - 1$ and creating one fermionic quantum $n_f \to n_f - 1$ or vice versa. We illustrate the annihilating (supersymmetric) charges (generators) for SUSY oscillator by letting $\omega_B = \omega_F = \omega$ as $\hat{Q} = \sqrt{\omega}\left(\hat{a}^{\dagger}\hat{b}^{\dagger}\right)$ & $\hat{Q}^{\dagger} = \sqrt{\omega}\left(\hat{b}^{\dagger}\hat{a}\right)$ …(1)

where $\hat{Q}\,\&\,\hat{Q}^{\dagger}$ satisfies

$$\left[\hat{Q},\hat{N}_B\right]=0 \quad , \quad \left[\hat{Q}^{\dagger},\hat{N}_B\right]=\hat{Q}^{\dagger} \quad , \quad \left[\hat{Q},\hat{N}_F\right]=0 \quad \& \quad \left[\hat{Q}^{\dagger},\hat{N}_F\right]=-\hat{Q}^{\dagger} \qquad \ldots (2a)$$

and

$$\hat{Q}\left|n_B,n_F\right\rangle = \sqrt{(n_B+1)}\,\left|n_B+1,n_F-1\right\rangle \qquad \text{if } n_F=1$$

$$= 0 \qquad \text{if } n_F=0$$

along with

$$\hat{Q}_{+}\left|n_B,n_F\right\rangle=\frac{1}{\sqrt{n_B}}\left|n_B-1,n_F+1\right\rangle \qquad \text{if } n_B\neq 0, n_F=0$$

$$=0 \qquad \text{if } n_B=0\, n_F=1. \qquad \ldots(2b)$$

Thus energy states $\left|n_B+1,n_F-1\right\rangle$ and $\left|n_B-1,n_F+1\right\rangle$ are degenerate in energy with the state $\left|n_B,n_F\right\rangle$. So that Hamiltonian $\hat{H}$ becomes

$$\hat{H} = \left\{\hat{Q}^{\dagger},\hat{Q}\right\} = \omega\left\{\hat{a}^{\dagger}\hat{b}\hat{b}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{b}\right\} = \omega\left\{\hat{a}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{b}\right\}$$

$$=\hat{H}_{osc}+\hat{H}_{spin}=\hat{H}_{Boson}+\hat{H}_{Fermion}=\omega\left(\hat{N}_B+\hat{N}_F\right), \qquad \ldots.(2c)$$

and

$$\left[\hat{H}\ ,\hat{Q}\ \right]=\left[\hat{H}\ ,\hat{Q}^{\ \dagger}\ \right]=\ 0\ . \qquad \ldots(2d)$$

Energy eigenvalue is $E=\omega\left(n_B+n_F\right)=\omega n$ …(3)

$n_F=0,1\ \&\ n_B=0,1,2,3...............$

In terms of quaternion component the total oscillator is written as

$$\hat{H}=\hat{H}_B+\hat{H}_F=\frac{\omega}{6}\left(\hat{a}_0^{\ 2}+\hat{a}_1^{\ 2}+\hat{a}_2^{\ 2}+\hat{a}_3^{\ 2}+2\hat{b}_0\left(\mathrm{e}_1\hat{b}_1+\mathrm{e}_2\hat{b}_2+\mathrm{e}_3\hat{b}_3\right)\right). \qquad \ldots(4)$$

which can be visualized analogously to the following expression of harmonic oscillator in one dimensional form [5] i.e.

$$\hat{H}=\left(-\frac{d^2}{dx^2}+\frac{x^2}{4}\right)-\frac{1}{2}\left[\psi,\psi^{\dagger}\right] \qquad \ldots(5)$$

where

$$\hat{H}_B=\left(-\frac{d^2}{dx^2}+\frac{x^2}{4}\right) \quad , \quad \hat{H}_F=-\frac{1}{2}\left[\psi,\psi^{\dagger}\right].$$

Here the term $2\hat{b}_0\left(\mathrm{e}_1\hat{b}_1+\mathrm{e}_2\hat{b}_2+\mathrm{e}_3\hat{b}_3\right)$ given in equation (4) removes the zero point energy. But non-linear.in nature hence in the general case we can write susy Hamiltonian in the form

$$\hat{H}=\left(-\frac{d^2}{dx^2}+w^2\right)-\left[\psi,\psi^{\dagger}\right]w'. \qquad \ldots(6)$$

In order to write the explicit form of a general supersymmetric harmonic oscillator Hamiltonian in three dimensional representation and accordingly to visualize the present theory of quaternionic harmonic oscillator in three dimension ,we may substitute the following relations between the operators;

$$\hat{a}_0^{\ 2}=-\frac{6}{\omega}W^2\ , \quad \hat{a}_1^{\ 2}=-\frac{6}{\omega}\frac{d^2}{dx^2},\ \hat{a}_2^{\ 2}=-\frac{6}{\omega}\frac{d^2}{dy^2},\quad \hat{a}_3^{\ 2}=-\frac{6}{\omega}\frac{d^2}{dz^2}, \qquad \ldots(7a)$$

$$\hat{b}_0\hat{b}_1=-\sqrt{\frac{\omega}{3}}\sigma_3W'(x),\quad \hat{b}_0\hat{b}_2=-\sqrt{\frac{\omega}{3}}\sigma_3W'(y),\quad \hat{b}_0\hat{b}_3=-\sqrt{\frac{\omega}{3}}\sigma_3W'(y) \qquad \ldots(7b)$$

As such the SUSY Harmonic oscillator becomes

$$\hat{H}=\left(-\frac{d^2}{dx^2}-\frac{d^2}{dy^2}-\frac{d^2}{dz^2}-W^2\right)-\sigma_3\left\{\mathrm{e}_1W'(x)+\mathrm{e}_2W'(y)+\mathrm{e}_3W'(z)\right\} \qquad \ldots(8)$$

This is three-dimensional representation of quaternionic harmonic oscillator. and can be reduced further to ther case of one dimension only by setting $Y = Z = 0$ *i.e.* one-dimensional harmonic oscillator is

$$\hat{H} = \left(-\frac{d^2}{dx^2} - w^2\right) - \sigma_3\left(\mathrm{e}_2 W'(x)\right) \quad = \left(-\frac{d^2}{dx^2} - w^2\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\left(\mathrm{e}_2 W'(x)\right) \qquad \ldots(9)$$

$$= \begin{bmatrix}\left(-\frac{d^2}{dx^2} - w^2 - \mathrm{e}_2 W'(x)\right) & 0\\ 0 & \left(-\frac{d^2}{dx^2} - w^2 + \mathrm{e}_2 W'(x)\right)\end{bmatrix} \quad = \begin{bmatrix}\hat{H}_+ & 0\\ 0 & \hat{H}_-\end{bmatrix} \qquad \ldots(10)$$

where $\hat{H}_+$ and $\hat{H}_-$ are denoted as quaternionic superpartner Hamiltonians i.e.

$$\hat{H} = \begin{bmatrix}AA^\dagger & 0\\ 0 & A^\dagger A\end{bmatrix} \qquad \ldots(11)$$

where A is generalized (combined bosonic and fermionic i.e. supersymmetric) annihilation operator and $A^\dagger$ is generalized creation operator, and given by

$$A = W(x) + e_2\frac{d}{dx} \qquad \& \qquad A^\dagger = -W(x) + e_2\frac{d}{dx} \qquad \ldots(12)$$

The supercharges given by equation (1) may thus be represented by

$$\hat{a} = \frac{1}{\sqrt{2\omega}}(\omega\hat{q} - e_1\hat{p}) \;\; \& \;\; \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}}(\omega\hat{q} + e_1\hat{p})$$

respectively for bosonic and fermionic operators along with the expression

$$\hat{P} = \frac{1}{e_1}\sqrt{\frac{\omega_b}{2}}\left(-\hat{a}^\dagger + \hat{a}\right) = \sqrt{\frac{\omega_b}{3}}(-\hat{a}_1 + e_3\hat{a}_2 - e_2\hat{a}_3) \;\; \& \;\; \hat{q} = \sqrt{\frac{2}{\omega_b}}\left(\hat{a}^\dagger + \hat{a}\right) \quad = \sqrt{\frac{3}{\omega_b}}\hat{a}_0$$

for position and momentum operators and accordingly we may thus obtain the supersymmetric Hamiltonian operator given by equation (11). Returning to equation (3), the eigen state is described as $\left|n_B, n_F\right\rangle$ and ground state as $\left|0,0\right\rangle$ so that

$$\hat{\mathrm{H}}\left|n_B, n_F\right\rangle = \mathrm{E}_{\,n_\mathrm{B}, n_\mathrm{F}}\left|n_\mathrm{B}, n_\mathrm{F}\right\rangle \quad , \quad n_\mathrm{B} = 0,1,2,3, \quad \& \quad n_\mathrm{F} = 0 \text{ or } 1. \qquad \ldots(13)$$

We also have

$$\hat{Q}\left|n, 1\right\rangle = \sqrt{n+1}\left|n+1, 0\right\rangle \qquad \& \qquad \hat{Q}^\dagger\left|n+1, 0\right\rangle = \sqrt{n+1}\left|n, 1\right\rangle \qquad \ldots(14)$$

These supercharges represents conversion of a fermionic state to a bosonic state and bosonic state to fermionic state

$$\hat{Q}^\dagger\left|\text{boson}\right\rangle = \left|\text{fermion}\right\rangle \quad , \quad \hat{Q}\left|\text{fermion}\right\rangle = \left|\text{boson}\right\rangle. \qquad ..(15)$$

Equation (2c) is the direct analogous of following equations of super symmetry

$$\{\hat{Q}_\alpha{}^\dagger, \hat{Q}_\beta\} = \mathrm{P}^\mu(\sigma_\mu)_{\alpha\beta} \qquad \text{and} \qquad \left[\hat{H}, \hat{Q}_\alpha\right] = 0 \qquad \ldots(16)$$

For $\mu = 0$ and $\alpha = \beta = 1$. Supercharges always commute with usual Hamiltonian. Thus the anticommuting charges in quaternion formalism combine to form the generators of time translation, namely the Hamiltonian $\hat{H}$. The ground state of this system is the state $\left|0\right\rangle_{osc}\left|0\right\rangle_{spin}$ or $\left|0\right\rangle_{boson}\left|0\right\rangle_{fermion} = \left|0,0\right\rangle$, where both bosonic and fermionic degrees of freedom are in the lowest energy state. This state is unique one and satiafies

$$\hat{Q}\left|0,0\right\rangle = \hat{Q}^\dagger\left|0,0\right\rangle = 0. \qquad \ldots(17)$$

As such we may calculate the energy of supersymmetric harmonic oscillator from equation (2) i.e.

$$\hat{H}\left|0,0\right\rangle = \omega\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\right)\left|0,0\right\rangle = 0 \qquad \ldots(18)$$

which shows that SUSY does not break for ground state, and we have the higher energy states in the following manner

$$\hat{H}\left|1,0\right\rangle = \omega\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\right)\left|1,0\right\rangle = \omega \quad \& \quad \hat{H}\left|0,1\right\rangle = \omega\left(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\right)\left|0,1\right\rangle = \omega \qquad \ldots(19)$$

and accordingly

$$\hat{H}\left|1,0\right\rangle = E\left|1,0\right\rangle = \omega \quad , \quad \hat{H}\left|2,0\right\rangle = E\left|1,1\right\rangle = 2\omega \;\;\&\;\; \hat{H}\left|3,0\right\rangle = E\left|2,1\right\rangle = 3\omega \qquad \ldots(20)$$

and so on. It shows that the energy states (1,0) and (0,1) are degenerate states. Similarly (2,0) and (1,1), (3,0), (2,1) and (1,2) are also degenerate. As such the excited states form a tower of degenerate levels (table) with energy $\left(n+\frac{1}{2}\right)\eta\omega \pm \frac{1}{2}\eta\omega$, where the sign of the second term is determined by whether the spin $\frac{1}{2}$ state is $\left|1\right\rangle$ (plus) or $\left|0\right\rangle$ (minus) . Illustration as follows

Energy	State	
	Boson	Fermion
0	$\left\|0,0\right\rangle$	
ω	$\left\|1,0\right\rangle$	$\left\|0,1\right\rangle$
2ω	$\left\|2,0\right\rangle$	$\left\|1,1\right\rangle$
3ω	$\left\|3,0\right\rangle$	$\left\|2,1\right\rangle$
4ω . .	$\left\|4,0\right\rangle$	$\left\|3,1\right\rangle$

The tower of states describes the boson fermion degenracy for exact supersymmetry. The bosonic state $\left|n+1,0\right\rangle$ (called bosonic in field theory analogy because they contain no fermions) have the same energy as their fermion partner in $\left|n,1\right\rangle$. Thus the quaternion reformation of a super symmetry gives rise to a simple representation of super symmetry in quantum mechanics. It is however, trivial since it describes non-interacting boson (oscillator) and fermions (spin $\left(-\frac{1}{2}\right)$ particles).

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